The Lovász Number of Random Graphs

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Abstract. We study the Lovász number ϑ along with two further SDP relaxations $\vartheta_{1/2}$, ϑ_2 of the independence number and the corresponding relaxations $\bar{\vartheta}$, $\bar{\vartheta}_{1/2}$, $\bar{\vartheta}_2$ of the chromatic number on random graphs $G_{n,p}$. We prove that $\vartheta, \vartheta_{1/2}, \vartheta_2(G_{n,p})$ are concentrated about their means, and that $\bar{\vartheta}, \bar{\vartheta}_{1/2}, \bar{\vartheta}_2(G_{n,p})$ in the case $p < n^{-1/2-\varepsilon}$ are concentrated in intervals of constant length. Moreover, extending a result of Juhász [27], we show that $\vartheta, \vartheta_{1/2}, \vartheta_2(G_{n,p}) = \Theta(\sqrt{n/p})$ and that $\bar{\vartheta}, \bar{\vartheta}_{1/2}, \bar{\vartheta}_2(G_{n,p}) = \Theta(\sqrt{np})$ for $c_0/n \leq p \leq 1/2$. As an application, we give an improved algorithm for approximating the independence number of $G_{n,p}$ in polynomial expected time, thereby extending a result of Krivelevich and Vu [33]. We also improve on the analysis of an algorithm of Krivelevich [30] for deciding whether $G_{n,p}$ is k-colorable.

Topics and key words: Lovász number, vector chromatic number, random graphs, maximum independent set problem, graph coloring

1 Introduction and Results

Given a graph $G = (V, E)$, let $\alpha(G)$ be the independence number, let $\omega(G)$ be the clique number, and let $\chi(G)$ be the chromatic number of G. Further, let $\bar{G}$ signify the complement of G. Since it is NP-hard to compute any of $\alpha(G)$, $\omega(G)$ or $\chi(G)$, it is remarkable that there exists an efficiently computable function $\vartheta(G)$ that is "sandwiched" between $\alpha(G)$ and $\chi(\bar{G})$, i.e. $\alpha(G) \leq \vartheta(G) \leq \chi(\bar{G})$. Passing to complements, and letting $\bar{\vartheta}(G) = \vartheta(\bar{G})$, we have $\omega(G) \leq \bar{\vartheta}(G) \leq \chi(G)$. The function ϑ was introduced by Lovász [35], and is called the *Lovász number* of G (cf. also [29]). The Lovász number can be seen as a semidefinite programming ("SDP") relaxation of the independence number, and is therefore comptable in polynomial time within any precision [24].

Though $\vartheta(G)$ is sandwiched between $\alpha(G)$ and $\chi(\bar{G})$, Feige [12] proved that the gap between $\alpha(G)$ and $\vartheta(G)$ or between $\chi(\bar{G})$ and $\vartheta(G)$ can be as large as $n^{1-\varepsilon}$, $\varepsilon > 0$. Indeed, unless NP=coRP, none of $\alpha(G)$, $\omega(G)$, $\chi(G)$ can be approximated within a factor of $n^{1-\varepsilon}$, $\varepsilon > 0$, in polynomial time [25,14]. However, though there exist graphs G such that $\vartheta(G)$ is not a good approximation of $\alpha(G)$ (or $\bar{\vartheta}(G)$ of $\chi(G)$), it might be the case that the Lovász number performs well on "average" instances. In fact, several algorithms for random and semirandom graph problems are based on computing ϑ [6,7,8,13,15]. Therefore, the aim of this paper is to study the Lovász number of random graphs more thoroughly.

The standard model of a random graph is the binomial model $G_{n,p}$, pioneered by Erdős and Renyi. We let $0 < p = p(n) < 1$ be a number that may depend on n. Let $V = \{1, \ldots, n\}$. Then the random graph $G_{n,p}$ is obtained by including each of the $\binom{n}{2}$ possible edges $\{v, w\}$, $v, w \in V$, with probability p independently. Though $G_{n,p}$ may fail to model some types of input instances appropriately, both the combinatorial structure and the algorithmic theory of $G_{n,p}$ are of fundamental interest [4,26,19]. We say that $G_{n,p}$ has some property A *with high probability* (whp.), if $\lim_{n\to\infty} \mathrm{P}(G_{n,p} \text{ has property } A) = 1$.

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In addition to the Lovász number, we also address two further natural SDP relaxations $\vartheta_{1/2}$, ϑ_2 of α (cf. [39]) on random graphs. These relaxations satisfy $\alpha(G) \leq \vartheta_{1/2}(G) \leq \vartheta(G) \leq \vartheta_2(G) \leq \chi(\bar{G})$, for all G, i.e. $\vartheta_{1/2}$ is the strongest relaxation of α among $\vartheta_{1/2}, \vartheta, \vartheta_2$. Passing to complements, and setting $\bar{\vartheta}_i(G) = \vartheta_i(\bar{G})$ $(i = 1/2, 2)$, one gets $\omega(G) \leq \bar{\vartheta}_{1/2}(G) \leq \bar{\vartheta}(G) \leq \bar{\vartheta}_2(G) \leq \chi(G)$, i.e. $\bar{\vartheta}_2$ is the strongest relaxation of χ. The relaxation $\bar{\vartheta}_{1/2}(G)$ coincides with the well-known *vector chromatic number* $\boldsymbol{\chi}(G)$ of Karger, Motwani, and Sudan [28].

The concentration of $\boldsymbol{\vartheta}$, $\bar{\boldsymbol{\vartheta}}$, etc. Facing a real-valued random variable $X(G_{n,p})$, there are two obvious questions to ask.

1. What is the mean of $X(G_{n,p})$?
2. Is $X(G_{n,p})$ concentrated about its mean?

The main contributions of this paper are concentration results on ϑ, $\bar{\vartheta}$ etc., i.e. concern the second question. Such results are important for instance in the design of algorithms with a polynomial expected running time. First, we show that the probability that $\vartheta(G_{n,p})$, $\vartheta_{1/2}(G_{n,p})$, or $\vartheta_2(G_{n,p})$ is far from its median is exponentially small.

Theorem 1. *Suppose that $p \leq 0.99$, and that $n \geq n_0$ for a certain constant $n_0 > 0$. Let m be a median of $\vartheta(G_{n,p})$, i.e. $\mathrm{P}(\vartheta(G_{n,p}) \leq m) \geq 1/2$ and $\mathrm{P}(\vartheta(G_{n,p}) \geq m) \geq 1/2$.*

i. Let $\xi \geq \max\{10, m^{1/2}\}$. Then $\mathrm{P}(\vartheta(G_{n,p}) \geq m + \xi) \leq 30\exp(-\xi^2/(5m + 10\xi))$.
ii. Let $\xi > 10$. Then $\mathrm{P}(\vartheta(G_{n,p}) \leq m - \xi) \leq 3\exp(-\xi^2/10m)$.

The same holds with ϑ replaced by $\vartheta_{1/2}$ or by ϑ_2.

Up to the constants involved, the right hand sides of the bounds in Thm. 1 are similar to well-known bounds on the tails of the binomial distribution (e.g. [26, p. 26]). The proof of Thm. 1 is based on Talagrand's inequality.

A remarkable fact concerning the chromatic number of sparse random graphs $G_{n,p}$, $p \leq n^{-\varepsilon-1/2}$, is that $\chi(G_{n,p})$ is concentrated in an interval of constant length. Indeed, Shamir and Spencer [38] proved that there is a function $u = u(n,p)$ such that in the case $p = n^{-\beta}$, $1/2 < \beta < 1$, we have $\mathrm{P}(u \leq \chi(G_{n,p}) \leq u + \lceil(2\beta+1)/(2\beta-1)\rceil) = 1 - o(1)$. Furthermore, Łuczak [36] showed that in the case $5/6 < \beta < 1$, the chromatic number is concentrated in width one, which is best possible. In fact, Alon and Krivelevich [2] could prove that two point concentration holds for the entire range $p = n^{-\beta}$, $1/2 < \beta < 1$. The two following theorems state similar results as given by Shamir and Spencer and by Łuczak for the relaxations $\bar{\vartheta}_{1/2}(G_{n,p})$, $\bar{\vartheta}(G_{n,p})$, and $\bar{\vartheta}_2(G_{n,p})$ of the chromatic number.

Theorem 2. *Suppose that $c_0/n \leq p \leq n^{-\beta}$ for some large constant $c_0 > 0$ and some number $1/2 < \beta < 1$. Then $\bar{\vartheta}_{1/2}(G_{n,p})$, $\bar{\vartheta}(G_{n,p})$, $\bar{\vartheta}_2(G_{n,p})$ are concentrated in width $s = \frac{2}{2\beta-1} + o(1)$, i.e. there exist numbers u, u', u'' depending on n and p such that whp.*

$$u \leq \bar{\vartheta}_{1/2}(G_{n,p}) \leq u + s, \;\; u' \leq \bar{\vartheta}(G_{n,p}) \leq u' + s, \;\; \textit{and } u'' \leq \bar{\vartheta}_{1/2}(G_{n,p}) \leq u'' + s.$$

Theorem 3. *Suppose that $c_0/n < p \leq n^{-5/6-\delta}$ for some large constant c_0 and some $\delta > 0$. Then $\bar{\vartheta}_{1/2}(G_{n,p})$, $\bar{\vartheta}(G_{n,p})$, and $\bar{\vartheta}_2(G_{n,p})$ are concentrated in width 1.*

In contrast to the chromatic number, $\bar{\vartheta}_{1/2}$, $\bar{\vartheta}$, and $\bar{\vartheta}_2$ need not be integral. Therefore, the above results do *not* imply that $\bar{\vartheta}_{1/2}(G_{n,p})$, $\bar{\vartheta}(G_{n,p})$, $\bar{\vartheta}_2(G_{n,p})$ are concentrated on a constant number of points. The proofs of Thms. 2 and 3 are given in Sec. 3.2.

The probable value of $\vartheta(G_{n,p})$, $\bar{\vartheta}(G_{n,p})$, etc. Concerning the probable value of $\vartheta(G_{n,p})$ and $\bar{\vartheta}(G_{n,p})$, Juhász [27] gave the following partial answer: If $\ln(n)^6/n \ll p \leq 1/2$, then with high probability we have $\vartheta(G_{n,p}) = \Theta(\sqrt{n/p})$ and $\bar{\vartheta}(G_{n,p}) = \Theta(\sqrt{np})$. However, we shall indicate in Sec. 4 that Juhász's proof fails in the case of sparse random graphs (e.g. $np = O(1)$). Making use of the above concentration results on ϑ, $\bar{\vartheta}$ etc., we can compute the probable value not only of $\vartheta(G_{n,p})$ and $\bar{\vartheta}(G_{n,p})$, but also of $\vartheta_i(G_{n,p})$ and $\bar{\vartheta}_i(G_{n,p})$, $i = 1/2, 2$, for essentially the entire range of edge probabilities p. To the best of the author's knowledge, no previous results concerning $\vartheta_i(G_{n,p})$ and $\bar{\vartheta}_i(G_{n,p})$, $i = 1/2, 2$, occur in the literature. Note that we only need to consider edge probabilities $p \leq 1/2$, because $G_{n,1-p} = \bar{G}_{n,p}$.

Theorem 4. *Suppose that $c_0/n \leq p \leq 1/2$ for some large constant $c_0 > 0$. Then there exist constants $c_1, c_2, c_3, c_4 > 0$ such that*

$$\begin{aligned} c_1\sqrt{n/p} \leq \vartheta_{1/2}(G_{n,p}) \leq \vartheta(G_{n,p}) \leq \vartheta_2(G_{n,p}) \leq c_2\sqrt{n/p} \\ \textit{and} \quad c_3\sqrt{np} \leq \bar{\vartheta}_{1/2}(G_{n,p}) \leq \bar{\vartheta}(G_{n,p}) \leq \bar{\vartheta}_2(G_{n,p}) \leq c_4\sqrt{np} \end{aligned} \tag{1}$$

with high probability. More precisely,

$$\mathrm{P}(c_3\sqrt{np} \leq \bar{\vartheta}_{1/2}(G_{n,p}) \leq \bar{\vartheta}(G_{n,p}) \leq \bar{\vartheta}_2(G_{n,p})) \geq 1 - \exp(-n). \tag{2}$$

Assume that $c_0/n \leq p = o(1)$. Then $\alpha(G_{n,p}) \sim 2\ln(np)/p$ and $\chi(G_{n,p}) \sim np/(2\ln(np))$ whp. (cf. [26]). Hence, Thm. 4 shows that $\vartheta_2(G_{n,p})$ (resp. $\bar{\vartheta}_{1/2}(G_{n,p})$) approximates $\alpha(G_{n,p})$ (resp. $\chi(G_{n,p})$) within a factor of $O(\sqrt{np})$. In fact, if $np = O(1)$, then we get a constant factor approximation. On the other hand, as $\alpha(G_{n,1/2}) \sim 2\log_2(n)$ and $\chi(G_{n,1/2}) \sim n/(2\log_2(n))$, in the random graph $G = G_{n,1/2}$ the gap between $\vartheta_{1/2}(G)$ (resp. $\bar{\vartheta}_2(G)$) and $\alpha(G)$ (resp. $\chi(G)$) is as large as $n^{1/2-\varepsilon}$ whp. Our estimate on the probable value of the vector chromatic number $\bar{\vartheta}_{1/2}(G_{n,p})$ in Thm. 4 answers a question of Krivelevich [30].

As a consequence of the upper bound on $\bar{\vartheta}_2(G_{n,p})$ in Thm. 4, we obtain a lower bound on the probable value of the SDP relaxation SDP_k of MAX k-CUT due to Frieze and Jerrum [18].

Corollary 5. *Let $k \geq 2$ be an integer. Suppose that $c_0k^2/n \leq p \leq 1/2$. Then whp. we have $\mathrm{SDP}_k(G_{n,p}) \geq \left(1 - \frac{1}{k}\right)\binom{n}{2}p + c_1n^{3/2}p^{1/2}$ for some constant $c_1 > 0$.*

Corollary 5 complements an upper bound on $\mathrm{SDP}_k(G_{n,p})$ due to Coja-Oghlan, Moore, and Sanwalani [9], who proved that there is a constant $c_2 > 0$ such that $\mathrm{SDP}_k(G_{n,p}) \leq (1 - 1/k)\binom{n}{2}p + c_2n^{3/2}p^{1/2}$ whp. In contrast, the weight $\mathrm{MC}_k(G_{n,p})$ of a MAX k-CUT of $G_{n,p}$ is at most

$$\left(1 - \frac{1}{k}\right)\binom{n}{2}p + \sqrt{\frac{\ln(k)}{k}} \cdot n^{3/2}p^{1/2}$$

whp. (cf. [9]). Thus, Cor. 5 shows that for large k there is a moderate gap between $\mathrm{SDP}_k(G_{n,p})$ and $\mathrm{MC}_k(G_{n,p})$.

Finally, let us consider the *random regular graph* $G_{n,r}$, i.e. an r-regular graph of order n chosen uniformly at random.

Theorem 6. *Let c_0 be a sufficiently large constant, and let $c_0 \leq r = o(n^{1/4})$. There are constants $c_1, c_2 > 0$ such that whp. the random regular graph $G_{n,r}$ satisfies*

$$c_1n/\sqrt{r} \leq \vartheta_{1/2}(G_{n,r}) \leq \vartheta(G_{n,r}) \leq \vartheta_2(G_{n,r}) \leq c_2n/\sqrt{r}.$$

Moreover, there is a constant $c_3 > 0$ such that in the case $c_0 \leq r = o(n^{1/2})$ we have

$$\mathrm{P}(c_3\sqrt{r} \leq \bar{\vartheta}_{1/2}(G_{n,r}) \leq \bar{\vartheta}(G_{n,r}) \leq \bar{\vartheta}_2(G_{n,r})) \geq 1 - \exp(-n). \tag{3}$$

Algorithmic applications. There are two types of algorithms for NP-hard random graph problems. First, there are *heuristics* that *always* run in polynomial time, and *almost always* output a good solution. On the other hand, there are algorithms that guarantee some approximation ratio on *any* input instance, and which have a polynomial *expected* running time when applied to $G_{n,p}$ (cf. [11]). Here we say that an algorithm A runs in polynomial expected time if there is a constant $l > 0$ such that $\sum_G R_A(G)\,\mathrm{P}(G_{n,p} = G) = O(n^l)$, where $R_A(G)$ is the running time of A on input G and the sum ranges over all graphs G of order n. In this paper, we are concerned with algorithms with a polynomial expected running time.

First, we consider the maximum independent set problem in random graphs. Krivelevich and Vu [33] gave an algorithm that in the case $p \gg n^{-1/2}$ approximates the independence number of $G_{n,p}$ in polynomial expected time within a factor of $O(\sqrt{np}/\ln(np))$. Moreover, they ask whether a similar algorithm exists for smaller values of p. A first answer was obtained by Coja-Oghlan and Taraz [7], who gave an $O(\sqrt{np}/\ln(np))$-approximative algorithm for the case $p \gg \ln(n)^6/n$. Using Thms. 1 and 4, we can improve on the analysis given in [7], thereby answering the question of Krivelevich and Vu in the affirmative.

Theorem 7. *Suppose that $c_0/n \le p \le 1/2$. There exists an algorithm `ApproxMIS` that for any input graph G outputs an independent set of size at least $\alpha(G)\ln(np)/(c_1\sqrt{np})$, and which applied to $G_{n,p}$ runs in polynomial expected time. Here $c_0, c_1 > 0$ denote constants.*

As a second application, we give an algorithm for deciding within polynomial expected time whether the input graph is k-colorable. Instead of $G_{n,p}$, we shall even consider the *semirandom model* $G^+_{n,p}$ that allows for an adversary to add edges to the random graph. More precisely, the semirandom graph $G^+_{n,p}$ is constructed in two steps as follows. First, a random graph $G_0 = G_{n,p}$ is chosen. Then, an adversary completes the instance $G = G^+_{n,p}$ by adding arbitrary edges to G_0. We say that *the expected running time of an algorithm $\mathcal{A}$ is polynomial over $G^+_{n,p}$*, if there is some constant l such that the expected running time of $\mathcal{A}$ is $O(n^l)$ regardless of the behavior of the adversary.

Theorem 8. *Suppose that $k = o(\sqrt{n})$, and that $p \ge c_0k^2/n$, for some constant $c_0 > 0$. There exists an algorithm $\mathtt{Decide}_k$ that for any input graph G decides whether G is k-colorable, and that applied to $G^+_{n,p}$ has a polynomial expected running time.*

The algorithm $\mathtt{Decide}_k$ is essentially identical with Krivelevich's algorithm for deciding k-colorability in polynomial expected time [30]. However, the analysis given in [30] requires that $np \ge \exp(\Omega(k))$, whereas Thm. 8 only requires that np is quadratic in k. The improvement results from the fact that the analysis given in this paper relies on the asymptotics for $\bar{\vartheta}_{1/2}(G_{n,p})$ derived in Thm. 4 (instead of the concept of semi-colorings). Finally, we prove that our algorithm $\mathtt{Decide}_k$ also applies to random regular graphs $G_{n,r}$.

Theorem 9. *Suppose that $c_0k^2 \le r = o(n^{1/2})$ for some constant $c_0 > 0$. Then, applied to $G_{n,r}$, the algorithm $\mathtt{Decide}_k$ has polynomial expected running time.*

Organization. First we recall the definitions of ϑ, $\bar{\vartheta}$ etc. and prove some elementary facts in Sec. 2. Sec. 3 deals with the concentration results, and Sec. 4 contains the proofs of Thm. 4 and Cor. 5. In Sec. 5 we prove Thm. 6, and finally Sec. 6 is devoted to the algorithms.

Notation. Throughout we let $V = \{1, \ldots, n\}$. If $G = (V, E)$ is a graph, and $U \subset V$, then $N(U)$ is the neighborhood of U, i.e. set of all $v \in V$ such that there is $w \in U$ satisfying $\{v, w\} \in E$. Moreover, $A(G)$ is the adjacency matrix of G. By $\mathbf{1}$ we denote the vector with all entries equal to one in any dimension. Furthermore, J denotes a square matrix of any size with all entries equal to one. If M is a real symmetric $n \times n$-matrix, then $\lambda_1(M) \ge \cdots \lambda_n(M)$ signify the eigenvalues of M, and $\|M\| = \max\{\lambda_1(M), -\lambda_n(M)\}$ is the spectral radius of M. We let $\langle\cdot,\cdot\rangle$ denote the scalar product of vectors. By $c_0, c_1, \ldots$ we denote constants, i.e. numbers that are independent of n and p.

2 Preliminaries

In this section we recall the definitions of ϑ, $\vartheta_{1/2}$, ϑ_2, and provide some elementary facts which will be useful later. We let $G = (V, E)$ be a graph, and let $\bar{G}$ be the complement of G. Let $(v_1, \ldots, v_n)$ be an n-tuple of unit vectors in $\mathbf{R}^n$, and let $k > 1$. Then $(v_1, \ldots, v_n)$ is a *vector k-coloring* of G if $\langle v_i, v_j \rangle \leq -1/(k-1)$ for all edges $\{i, j\} \in E$. Furthermore, $(v_1, \ldots, v_n)$ is a *strict* vector k-coloring if $\langle v_i, v_j \rangle = -1/(k-1)$ for all $\{i, j\} \in E$. Finally, we say that $(v_1, \ldots, v_n)$ is a *rigid* vector k-coloring if $\langle v_i, v_j \rangle = -1/(k-1)$ for all $\{i, j\} \in E$ and $\langle v_i, v_j \rangle \geq -1/(k-1)$ for all $\{i, j\} \notin E$. Following [28,21,5], we define

$$\begin{aligned} \bar{\vartheta}_{1/2}(G) &= \inf\{k > 1 |\ G \text{ admits a vector } k\text{-coloring}\}, \\ \bar{\vartheta}(G) = \bar{\vartheta}_1(G) &= \inf\{k > 1 |\ G \text{ admits a strict vector } k\text{-coloring}\}, \\ \bar{\vartheta}_2(G) &= \inf\{k > 1 |\ G \text{ admits a rigid vector } k\text{-coloring}\}. \end{aligned} \tag{4}$$

Observe that $\bar{\vartheta}_{1/2}(G)$ is precisely the *vector chromatic number* introduced by Karger, Motwani, and Sudan [28]; $\bar{\vartheta}_2$ occurs in [21,39]. Further, we let $\vartheta_{1/2}(G) = \bar{\vartheta}_{1/2}(\bar{G})$, $\vartheta(G) = \vartheta_1(G) = \bar{\vartheta}(\bar{G})$, and $\vartheta_2(G) = \bar{\vartheta}_2(\bar{G})$. It is shown in [28] that the above definition of ϑ is equivalent with Lovász's original definition [35].

Proposition 10. *Let $G = (V, E)$ be a graph of order n, and let $S \subset V$. Let $G[S]$ denote the subgraph of G induced on S. Then $\vartheta_i(G) \leq \vartheta_i(G[S]) + \vartheta_i(G[V \setminus S])$, $i \in \{1/2, 1, 2\}$.*

Although the result may be known to specialists in the area, to the best of the author's knowledge it is not explicitly stated (or proved) in the literature. Therefore, we carry out the proof for $\bar{\vartheta}_2$; the same argument applies to $\bar{\vartheta}_{1/2}$ and $\bar{\vartheta}$.

Proof of Prop. 10. Let $k > \bar{\vartheta}_2(G[S])$ and let $l > \bar{\vartheta}_2(G[V \setminus S])$. Further, let $(a_v)_{v \in S}$ be a rigid vector k-coloring of $G[S]$, and let $(b_v)_{v \in V \setminus S}$ be a rigid vector l-coloring of $G[V \setminus S]$. Set

$$\alpha = \left(\frac{l}{(k+l)(k-1)} \right)^{1/2} \text{ and } \beta = \left(\frac{k}{(k+l)(l-1)} \right)^{1/2}.$$

Embedding the a_v's and b_w's into a high-dimensional space, we may assume that $a_v \perp b_w$ for all $v \in S$, $w \in V \setminus S$, and that there is a unit vector z such that $z \perp a_v$, $z \perp b_w$ for all v, w. Let

$$x_v = (1 + \alpha^2)^{-1/2}(a_v + \alpha z) \text{ and } x_w = (1 + \beta^2)^{-1/2}(b_w - \beta z) \qquad (v \in S, w \in V \setminus S).$$

Then $\|x_v\| = \|x_w\| = 1$. Moreover, if two vertices $v, v' \in S$ are adjacent, then

$$\langle x_v, x_{v'} \rangle = (1 + \alpha^2)^{-1} \left(\langle a_v, a_{v'} \rangle + \alpha^2 \right) = (1 + \alpha^2)^{-1} \left(-\frac{1}{k-1} + \alpha^2 \right) = -\frac{1}{k+l-1}.$$

Likewise, if $v, v' \in S$ are non-adjacent, then $\langle x_v, x_{v'} \rangle \geq -1/(k+l-1)$. Consequently, $(x_v)_{v \in S}$ is a rigid vector $(k+l)$-coloring of $G[S]$. Similarly, $(x_w)_{w \in V \setminus S}$ is a rigid vector $(k+l)$-coloring of $G[V \setminus S]$. Since $\langle x_v, x_w \rangle = -1/(k+l-1)$ for all $v \in S$, $w \in V \setminus S$, $(x_v)_{v \in V}$ is a rigid vector $(k+l)$-coloring of the entire graph G, thereby proving $\bar{\vartheta}_2(G) \leq k + l$. $\square$

In addition to Prop. 10, we will frequently make use of the well-known fact that

$$\omega(G) \leq \bar{\vartheta}_{1/2}(G) \leq \bar{\vartheta}(G) \leq \bar{\vartheta}_2(G) \leq \chi(G) \text{ and } \alpha(G) \leq \vartheta_{1/2}(G) \leq \vartheta(G) \leq \vartheta_2(G)$$

for all graphs $G = (V, E)$. The lower bounds $\omega(G) \leq \bar{\vartheta}_{1/2}(G)$, $\alpha(G) \leq \vartheta_{1/2}(G)$ are established in [28]. The upper bound $\bar{\vartheta}_2(G) \leq \chi(G)$ can be proved e.g. by decomposing V into $\chi(G)$ disjoint independent sets and applying Prop. 10. Moreover, it is obvious from the definitions that for any weak subgraph H of G we have

$$\bar{\vartheta}_i(H) \leq \bar{\vartheta}_i(G) \qquad (i \in \{1/2, 1, 2\}). \tag{5}$$

In addition to ϑ, $\vartheta_{1/2}$, and ϑ_2, we consider the following semidefinite relaxation of MAX k-CUT, due to Frieze and Jerrum [18]. Let G be a graph with adjacency matrix $A = A(G) = (a_{ij})_{i,j=1,\dots,n}$, and let $k \geq 2$. Then

$$\mathrm{SDP}_k(G) = \max \sum_{i<j} a_{ij} \frac{k-1}{k} (1 - \langle v_i, v_j \rangle) \text{ s.t. } \|v_i\| = 1,\ \langle v_i, v_j \rangle \geq -\frac{1}{k-1}, \tag{6}$$

where the max is taken over $v_1, \dots, v_n \in \mathbf{R}^n$, is an upper bound on the weight of a MAX k-CUT of G. In the case $k = 2$, we obtain the semidefinite relaxation SMC $= \mathrm{SDP}_2$ of MAX CUT invented by Goemans and Williamson [22]. In this case, the constraint $\langle v_i, v_j \rangle \geq -1/(2-1) = -1$ is void.

3 The Concentration Results

3.1 Proof of Theorem 1

The large deviation result for $\boldsymbol{\vartheta}$. In order to bound the probability that the Lovász number $\vartheta(G_{n,p})$ is far from its median, we shall apply the following version of *Talagrand's inequality* (cf. [26, p. 44]).

Theorem 11. *Let $\Lambda_1, \dots, \Lambda_N$ be probability spaces. Let $\Lambda = \Lambda_1 \times \dots \times \Lambda_N$. Let $A, B \subset \Lambda$ be measurable sets such that for some $t \geq 0$ the following condition is satisfied: For every $b \in B$ there is $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{R}^N \setminus \{0\}$ such that for all $a \in A$ we have*

$$\sum_{i:\, a_i \neq b_i} \alpha_i \geq t \left(\sum_{i=1}^N \alpha_i^2 \right)^{1/2},$$

where a_i (resp. b_i) denotes the i'th coordinate of a (resp. b). Then $\mathrm{P}(A)\mathrm{P}(B) \leq \exp(-t^2/4)$.

Let $G = (V, E)$ be a graph. We need the following equivalent characterization of $\vartheta(G)$. A tuple $(v_1, \dots, v_n)$ of vectors $v_i \in \mathbf{R}^d$ is called an *orthogonal labeling* of $\bar{G}$ if for any two vertices $i, j \in V$, $i \neq j$, with $\{i, j\} \in E$ we have $v_i \perp v_j$ (cf. [29]). Here $d > 0$ is any integer. Furthermore, the *cost* of a d-dimensional vector $a = {}^t(a_1, \dots, a_d)$ is

$$c(a) = \begin{cases} a_1^2 \|a\|^{-2} & \text{if } a \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $0 \leq c(a) \leq 1$, and we have

$$\vartheta(G) = \max \left\{ \sum_{i=1}^n c(v_i) |\ (v_1, \dots, v_n) \text{ is an orthogonal labelling of } \bar{G} \right\} \quad \text{(cf. [28,29]).} \tag{7}$$

The proof of Thm. 1 relies on the following lemma.

Lemma 12. *Let m be a median of $\vartheta(G_{n,p})$. Let $\vartheta_0 > 0$ be any number, and let $\xi \geq 10$. Then*

$$\mathrm{P}(m + \xi \leq \vartheta(G_{n,p}) \leq \vartheta_0) \leq 2 \exp(-\xi^2/(5\vartheta_0)).$$

Proof. For $i \geq 2$, let $\Lambda_i \in \{0,1\}^{i-1}$ consist of the first $i-1$ entries of the ith row of the adjacency matrix of $G_{n,p}$. Then $\Lambda_2, \dots, \Lambda_n$ are independent random variables, and Λ_i determines to which of the $i-1$ vertices $1, \dots, i-1$ vertex i is adjacent. Therefore, we can identify $G_{n,p}$ with the product space $\Lambda_2 \times \dots \times \Lambda_n$. Let $\pi_i : G_{n,p} = \Lambda_2 \times \dots \times \Lambda_n \to \Lambda_i$ be the ith projection. Let $A = \{G \in G_{n,p} |\ \vartheta(G) \leq m\}$ and $B = \{H \in G_{n,p} |\ m + \xi \leq \vartheta(H) \leq \vartheta_0\}$.

Let $H \in B$, and let $(b_1, \ldots, b_n)$ be an orthogonal labeling of $\bar{H}$ such that

$$m + \xi \leq \vartheta(H) = \sum_{i=1}^{n} c(b_i). \tag{8}$$

Set $\alpha_i = c(b_i)$, and $\alpha = (\alpha_2, \ldots, \alpha_n)$. As $0 \leq \alpha_i \leq 1$ for all i, we have

$$\sum_{i=2}^{n} \alpha_i^2 \leq \sum_{i=1}^{n} \alpha_i = \vartheta(H) \leq \vartheta_0. \tag{9}$$

Now let $G \in A$, set $a_1 = 0$, and let

$$a_i = \begin{cases} b_i \text{ if } \pi_i(G) = \pi_i(H) \\ 0 \text{ otherwise} \end{cases} \qquad \text{for } i = 2, \ldots, n.$$

We claim that $(a_1, \ldots, a_n)$ is an orthogonal labeling of $\bar{G}$. For if $i, j \in V$ are adjacent in G, and $i < j$, then we either have $\pi_j(G) = \pi_j(H)$ or $a_j = 0$. In the first case, i and j are adjacent in H, whence $\langle a_i, a_j \rangle = \langle b_i, b_j \rangle = 0$. Moreover, if $a_j = 0$, then obviously $a_i \perp a_j$. Thus, as $(a_1, \ldots, a_n)$ is an orthogonal labeling of $\bar{G}$, we have $\sum_{i=1}^{n} c(a_i) \leq \vartheta(G) \leq m$. Hence, Eq. (8) yields

$$\xi \leq c(b_1) + \sum_{i=2}^{n} c(b_i) - c(a_i) \leq 1 + \sum_{i:\, \pi_i(G) \neq \pi_i(H)} c(b_i) = 1 + \sum_{i:\, \pi_i(G) \neq \pi_i(H)} \alpha_i. \tag{10}$$

Set $t = (\xi - 1)/\sqrt{\vartheta_0}$. Then, by (9) and (10), for all $G \in A$ we have

$$\sum_{i:\, \pi_i(G) \neq \pi_i(H)} \alpha_i \geq t \left(\sum_{i=1}^{n} \alpha_i^2 \right)^{1/2}.$$

Consequently, Thm. 11 entails

$$\mathrm{P}(A)\mathrm{P}(m + \xi \leq \vartheta(G_{n,p}) \leq \vartheta_0) \leq \exp(-t^2/4) = \exp\left(-\frac{(\xi - 1)^2}{4\vartheta_0} \right) \leq \exp\left(-\frac{\xi^2}{5\vartheta_0} \right).$$

Hence, our assertion follows from the fact that $\mathrm{P}(A) \geq 1/2$. □

Proof of Thm. 1. By our assumption that $p \leq 0.99$, we have that $\vartheta(G_{n,p}) \geq \alpha(G_{n,p}) = \Omega(\ln(n))$ whp. Hence, we can choose n_0 large enough such that any median m of $\vartheta(G_{n,p})$ satisfies $m \geq c_0$ for some large constant c_0. As for the upper tail bound i.), by L. 12 we have

$$\begin{aligned} \mathrm{P}(m + \xi \leq \vartheta) &\leq \sum_{l=1}^{\infty} \mathrm{P}(m + l\xi \leq \vartheta \leq m + (l+1)\xi) \\ &\leq \sum_{l=1}^{\infty} 2 \exp\left(-\frac{l^2 \xi^2}{5(m + (l+1)\xi)} \right) \leq 2 \sum_{l=1}^{\infty} \exp\left(-\frac{l\xi^2}{5(m + 2\xi)} \right) \\ &\leq 30 \exp\left(-\frac{\xi^2}{5m + 10\xi} \right), \end{aligned}$$

as desired.

In order to prove the lower tail bound ii.), observe that by i.) we can choose c_0 large enough such that $\mathrm{P}(\vartheta(G_{n,p}) \geq 2m) < 1/6$, say. Consequently, $\mathrm{P}(m \leq \vartheta(G_{n,p}) \leq 2m) \geq 1/3$. Let $A = \{G |\ \vartheta(G) \leq m - \xi\}$ and $B = \{H |\ m \leq \vartheta(H) \leq 2m\}$. Then, a similar argument as in the proof of L. 12 yields

$$\mathrm{P}(A)\mathrm{P}(B) \leq \exp(-t^2/4) = \exp\left(-\frac{(\xi - 1)^2}{8m} \right) \leq \exp\left(-\frac{\xi^2}{10m} \right).$$

Thus, our assertion follows from the fact that $\mathrm{P}(B) \geq 1/3$. □

The large deviation result for $\vartheta_{1/2}$. To prove the bounds in Thm. 1 for $\vartheta_{1/2}$, we make use of a characterization of $\vartheta_{1/2}$ established in [23]. Let $(v_1, \ldots, v_n)$ be an assignment of vectors $v_1, \ldots, v_n \in \mathbf{R}^d$ to the vertices of $G = (V, E)$, where $d > 0$ is any integer. Let us call $(v_1, \ldots, v_n)$ a *strong orthogonal labeling* of $\bar{G}$ if $v_i \perp v_j$ whenever $\{i, j\} \in E$, and $\langle v_i, v_j \rangle \geq 0$ for all i, j. As shown in [23, pp. 51ff],

$$\vartheta_{1/2}(G) = \max \left\{ \sum_{i=1}^{n} c(v_i) | \ (v_1, \ldots, v_n) \text{ is a strong orthogonal labeling of } \bar{G} \right\}; \tag{11}$$

the proof goes along the lines of [29]. Using Eq. (11), the argument given for ϑ above carries over without essential changes and yields the proof of the tail bounds for $\vartheta_{1/2}$.

The large deviation result for ϑ_2. We shall establish a characterization of ϑ_2 that corresponds to the characterization (7) of ϑ, and which may be of independent interest. Let $G = (V, E)$ be a graph. If $x, y \in \mathbf{R}^d$, then we let $c(x, y) = \langle x, y \rangle^2 \|x\|^{-2} \|y\|^{-2}$, if $x, y \neq 0$, and $c(x, y) = 0$ otherwise. Moreover, we call a family $(v_0, \ldots, v_n)$ of vectors $v_i \in \mathbf{R}^d$ a *weak orthogonal labeling* of $\bar{G}$ if $\langle v_0, v_i \rangle \geq 0$ for all i, and $\langle v_i, v_j \rangle \leq 0$ if $\{i, j\} \in E$, $i, j = 1, \ldots, n$. Here d is any positive integer. Note that a weak orthogonal labeling consists of $n + 1 = \#V + 1$ vectors. We define

$$\vartheta_2'(G) = \max \left\{ \sum_{i=1}^{n} c(v_i, v_0) | \ (v_0, \ldots, v_n) \text{ is a weak orthogonal labeling of } G \right\}. \tag{12}$$

Lemma 13. *We have $\vartheta_2(G) = \vartheta_2'(G)$ for all graphs $G = (V, E)$.*

Proof. The following formulation of ϑ_2 as a semidefinite program has been given in [39]:

$$\vartheta_2(G) = \max \sum_{i,j=1}^{n} b_{ij} \text{ s.t. } b_{ij} \leq 0 \text{ for all } \{i, j\} \in E \text{ and } \sum_{i=1}^{n} b_{ii} = 1, \tag{13}$$

where the max is taken over all positive semidefinite matrices $B = (b_{ij})_{i,j}$. (One can prove Eq. (13) e.g. using a similar argument as given in [28] to prove that (4) is equivalent to Lovász's original definition of $\bar{\vartheta}$.) To prove that $\vartheta_2(G) \leq \vartheta_2'(G)$, let $B = (b_{ij})$ be a feasible matrix that maximizes (13). Since B is positive semidefinite, there are vectors $b_1, \ldots, b_n \in \mathbf{R}^n$ such that $b_{ij} = \langle b_i, b_j \rangle$. Let $b = \sum_{i=1}^{n} b_i$. Then $\langle b_k, b \rangle \geq 0$ for all k. For assume otherwise, and consider the matrix $B' = (b'_{ij})$, where $b'_{ij} = b_{ij}$ for $i, j \neq k$, $b'_{ki} = b'_{ik} = 0$ for $i \neq k$, and $b'_{kk} = b_{kk}$. Then B' is positive semidefinite, and is a feasible solution to (13). Consequently, our assumption $0 > \langle b_k, b \rangle = \sum_j b_{kj} = \sum_j b_{jk}$ entails

$$\sum_{i,j} b_{ij} = \vartheta_2(G) \geq \sum_{i,j} b'_{ij} = \sum_{i,j} b_{ij} - 2 \sum_{i \neq k} b_{ik} > \sum_{i,j} b_{ij},$$

a contradiction. Hence, $\langle b_k, b \rangle \geq 0$ for all k. Letting $v_0 = b/\|b\|$, $v_i = b_i/\|b_i\|$ if $b_i \neq 0$, and $v_i = 0$ otherwise $(i = 1, \ldots, n)$, we obtain a weak orthogonal labeling of $\bar{G}$ satisfying $\sum_{i=1}^{n} c(v_i, v_0) \geq \sum_{i,j} b_{ij} = \vartheta_2(G)$ (cf. the proof of Thm. 5 in [35]).

Conversely, let $(v_0, \ldots, v_n)$ be a weak orthogonal labeling of $\bar{G}$ such that $\sum_{i=1}^{n} c(v_i, v_0) = \vartheta_2'(G)$. We may assume that v_i either is a unit vector or is equal to zero for all i. Let

$$b_i = \vartheta_2'(G)^{-1/2} \langle v_0, v_i \rangle v_i,$$

set $b_{ij} = \langle b_i, b_j \rangle$, and $B = (b_{ij})_{i,j}$. Then B is positive semidefinite, and if $\{i, j\} \in E$, then $b_{ij} = \langle v_0, v_i \rangle \langle v_0, v_j \rangle \langle v_i, v_j \rangle / \vartheta_2'(G) \leq 0$, because $\langle v_0, v_i \rangle, \langle v_0, v_j \rangle \geq 0 \geq \langle v_i, v_j \rangle$. Moreover,

$\sum_{i=1}^n b_{ii} = \sum_{i=1}^n c(v_0, v_i)/\vartheta_2'(G) = 1$, whence B is a feasible solution to (13). Finally, to show that $\sum_{i,j} b_{ij} \geq \vartheta_2'(G)$, we adapt the argument used in [23] to prove (11): Let $M = \sum_{i=1}^n v_i{}^t v_i$. Then $\vartheta_2'(G) = \sum_i \langle v_0, v_i\rangle^2 = \langle Mv_0, v_0\rangle \leq \|Mv_0\|$. Consequently,

$$\vartheta_2'(G) \leq \|M(\vartheta_2'(G)^{-1/2}v_0)\|^2 = \|\sum_{i=1}^n \vartheta_2'(G)^{-1/2}\langle v_0, v_i\rangle v_i\|^2 = \sum_{i,j} b_{ij} \leq \vartheta_2(G),$$

thereby proving the lemma. □

Using the characterization (12) of ϑ_2, the arguments used to prove Thm. 1 for $\vartheta(G_{n,p})$ also apply to $\vartheta_2(G_{n,p})$.

3.2 Concentration of $\bar{\vartheta}_{1/2}$, $\bar{\vartheta}$, and $\bar{\vartheta}_2$ in Intervals of Constant Length

Though the proofs of Thms. 2 and 3 go along the lines of [36,38], we have to replace arguments concerning the chromatic number by arguments that apply to $\bar{\vartheta}_{1/2}$, $\bar{\vartheta}$, and $\bar{\vartheta}_2$. We shall demonstrate the proofs for $\bar{\vartheta}_2$, as this turns out to be the most demanding case. All arguments carry over to $\bar{\vartheta}_{1/2}$ and $\bar{\vartheta}$ immediately. We adapt a simplification of the argument given in [38] attributed to Frieze in [36].

Proof of Thm. 2. Let p and β be as in Thm. 2. The proof is based on the following large deviation result, which is a consequence of Azuma's inequality (cf. [26, p. 37]).

Lemma 14. *Suppose that $X : G_{n,p} \to \mathbf{R}$ is a random variable that satisfies the following conditions for all graphs $G = (V, E)$.*

- *For all $v \in V$ the following holds. Let $G^* = G + \{\{v, w\}|\ w \in V,\ w < v\}$, and let $G_* = G - \{\{v, w\}|\ w \in V,\ w < v\}$. Then $|X(G^*) - X(G_*)| \leq 1$.*
- *If H is a weak subgraph of G, then $X(H) \leq X(G)$.*

Then $\mathrm{P}(|X - \mathrm{E}(X)| > t\sqrt{n}) \leq 2\exp(-t^2/2)$.

Let $\omega = \omega(n)$ be a sequence tending to infinity slowly, e.g. $\omega(n) = \ln\ln(n)$. Furthermore, let

$$k = k(n, p) = \inf\{x > 0|\ \mathrm{P}(\bar{\vartheta}_2(G_{n,p}) \leq x) \geq \omega^{-1}\}. \tag{14}$$

For any graph $G = (V, E)$ let

$$Y(G) = Y_k(G) = \min\{\#U|\ U \subset V,\ \bar{\vartheta}_2(G - U) \leq k\}.$$

Then $\bar{\vartheta}_2(G) \leq k$ if and only if $Y(G) = 0$. Hence, $\mathrm{P}(Y = 0) \geq \omega^{-1}$. Moreover, by Prop. 10 and (5), the random variable Y satisfies the assumptions of L. 14. Therefore, letting $\mu = \mathrm{E}(Y)$, for any $\lambda > 0$ we have

$$\mathrm{P}(|Y(G_{n,p}) - \mu| \geq \lambda\sqrt{n}) \leq 2\exp(-\lambda^2/2). \tag{15}$$

We claim that $\mu \leq \sqrt{n}\omega$. For if $\mu > \sqrt{n}\omega$, then (15) yields

$$\omega^{-1} \leq \mathrm{P}(Y = 0) \leq \mathrm{P}(Y \leq \mu - \sqrt{n}\omega) \leq 2\exp(-\omega^2/2),$$

a contradiction. Thus, again by L. 14, $Y \leq 2\sqrt{n}\omega$ with high probability. The following lemma is implicit in [38] (cf. the proof of L. 8 in [38]).

Lemma 15. *Let $\delta > 0$. Whp. the random graph $G = G_{n,p}$ enjoys the following property. If $U \subset V$, $\#U \leq 2\sqrt{n}\omega$, then $\#E(G[U]) < \#Us/2$, where $s > \frac{2}{2\beta-1} + \delta$. Consequently, $\chi(G[U]) \leq s$,*

To conclude the proof of Thm. 2, let $G = G_{n,p}$, and suppose that there is some $U \subset V$, $\#U \leq 2\sqrt{n}\omega$, such that $\bar{\vartheta}_2(G - U) \leq k \leq \bar{\vartheta}_2(G)$. Since by L. 15 $\bar{\vartheta}_2(G[U]) \leq \chi(G[U]) \leq s$ whp., Prop. 10 entails that $k \leq \bar{\vartheta}_2(G) \leq k + s$ whp., thereby proving Thm. 2.

Proof of Thm. 3. Let $\omega = \omega(n) = (\ln\ln n)^{1/3}$ be a sequence tending to infinity slowly. By L. 15, the random graph $G = G_{n,p}$ admits no $U \subset V$, $\#U \leq \omega^3\sqrt{n}$, spanning more than $3(\#U - \varepsilon)/2$ edges whp., where $\varepsilon > 0$ is a small constant. Let k be defined as in (14). As shown in the proof of Thm. 2, whp. there is a set $U \subset V$, $\#U \leq \omega\sqrt{n}$, such that $\bar{\vartheta}_2(G - U) \leq k$. Following Łuczak [36], we let $U = U_0$, and construct a sequence $U_0, \ldots, U_m$ as follows. If there is no edge $\{v, w\} \in E$ with $v, w \in N(U_i) \setminus U_i$, then we let $m = i$ and finish. Otherwise, we let $U_{i+1} = U_i \cup \{v, w\}$ and continue. Then $m \leq m_0 = \omega^2\sqrt{n}$, because otherwise $\#U_{m_0} = (2 + o(1))\omega^2\sqrt{n}$ and $\#E(G[U_{m_0}]) \geq 3(1 - o(1))\#U_{m_0}/2$. Let $R = U_m$.

By L. 15, $\bar{\vartheta}_2(G[R]) \leq \chi(G[R]) \leq 3$. Furthermore, $I = N(R) \setminus R$ is an independent set. Let $G_1 = G[R \cup I]$, $S = V \setminus (R \cup I)$, and $G_2 = G[S \cup I]$. Then $\bar{\vartheta}_2(G_2) \leq k$, and $\bar{\vartheta}_2(G_1) \leq 4$. In order to prove that $\bar{\vartheta}_2(G) \leq k + 1$, we shall first construct a rigid vector $k + 1$-coloring of G_2 that assigns the same vector to all vertices in I. Thus, let $(x_v)_{v \in S \cup I}$ be a rigid vector k-coloring of G_2. Let x be a unit vector perpendicular to x_v for all $v \in S$. Moreover, let $\alpha = (k^2 - 1)^{-1/2}$, and set

$$y_v = \begin{cases} (\alpha^2 + 1)^{-1/2}(x_v - \alpha x) & \text{for } v \in S \\ x & \text{for } v \in I. \end{cases}$$

Then all y_v are unit vectors, and if $v \in S$, $w \in I$, then $\langle y_v, y_w \rangle = \langle y_v, x \rangle = -1/k$. Further, if $v, w \in S$ are adjacent in G_2, then

$$\langle y_v, y_w \rangle = \frac{1}{\alpha^2 + 1}\left(\langle x_v, x_w \rangle + \alpha^2\right) = \frac{1}{\alpha^2 + 1}\left(-\frac{1}{k - 1} + \alpha^2\right) = -\frac{1}{k}.$$

Likewise, if $v, w \in S$ are non-adjacent in G_2, then $\langle y_v, y_w \rangle \geq -1/k$, thereby proving that $(y_v)_{v \in S \cup I}$ is a rigid vector $(k + 1)$-coloring of G_2. In a similar manner, we can construct a rigid vector 4-coloring $(y'_v)_{v \in R \cup I}$ of G_1 that assigns the same vector x' to all vertices in I.

Applying a suitable orthogonal transformation if necessary, we may assume that $x = x'$. Let $l = \max\{4, k + 1\}$. Since $N(R) \subset R \cup I$, we obtain a rigid vector l-coloring $(z_v)_{v \in V}$ of G, where $z_v = y_v$ if $v \in S \cup I$, and $z_v = y'_v$ if $v \in R$. By the lower bound on $\bar{\vartheta}_2(G_{n,p})$ in Thm. 4 (which does not rely on Thm. 3 of course), choosing c_0 large enough we may assume that $k \geq 4$, whence $k \leq \bar{\vartheta}_2(G) \leq k + 1$.

4 The Probable Value of $\vartheta(G_{n,p})$, $\bar{\vartheta}(G_{n,p})$, etc.

In Sec. 4.1 we prove the lower bounds asserted in Thm. 4. These follow from results on the SDP relaxation of MAX CUT on random graphs due to Coja-Oghlan, Moore, and Sanwalani [9], and do not depend on the concentration results in the previous section. In Sec. 4.3 and 4.4 we prove the upper bounds on ϑ_2 and $\bar{\vartheta}_2$, which rely on Thm. 1, Thm. 3, and on a lemma on the spectrum of a certain auxiliary matrix given in Sec. 4.2. Finally, in Sec. 4.5 we prove Cor. 5.

4.1 The Lower Bound on $\bar{\vartheta}_{1/2}(G_{n,p})$

To bound $\bar{\vartheta}_{1/2}(G_{n,p})$ from below, we make use of an estimate on the probable value of the SDP relaxation SMC = SDP_2 of MAX CUT (cf. Sec. 2 for the definition). Combining Thms. 4 and 5 of [9] instantly yields the following bound on the probable value of $\mathrm{SMC}(G_{n,p})$.

Lemma 16. *Suppose that $c_0/n \leq p \leq 1 - c_0/n$ for some large constant $c_0 > 0$. There is a constant $\lambda > 0$ (independent of n, p) such that*

$$\mathrm{P}\left(\mathrm{SMC}(G_{n,p}) > \frac{1}{2}\binom{n}{2}p + \lambda n^{3/2}p^{1/2}(1 - p)^{1/2}\right) \leq \exp(-2n). \tag{16}$$

Let $G = (V, E)$ be a graph with adjacency matrix $A = (a_{ij})_{i,j=1,\dots,n}$. Let $v_1, \dots, v_n$ be a vector k-coloring of G, where $k = \bar{\vartheta}_{1/2}(G) \geq 2$. Then $\|v_i\| = 1$ for all i, and $\langle v_i, v_j \rangle \leq -1/(k-1)$ whenever $\{i, j\} \in E$. Therefore, we can interpret $v_1, \dots, v_n$ as a feasible solution to SMC, whence

$$\mathrm{SMC}(G) \geq \sum_{i<j} \frac{a_{ij}}{2}(1 - \langle v_i, v_j \rangle) \geq \#E\left(\frac{1}{2} + \frac{1}{k-1}\right) = \#E\left(\frac{1}{2} + \frac{1}{\bar{\vartheta}_{1/2}(G) - 1}\right). \qquad (17)$$

Let $c_0/n \leq p \leq 1 - c_0/n$ for some large constant $c_0 > 0$. By Chernoff bounds (cf. [26, p. 26]),

$$\mathrm{P}\left(\#E(G_{n,p}) < \binom{n}{2}p - 8n^{3/2}p^{1/2}(1-p)^{1/2}\right) \leq \exp(-2n). \qquad (18)$$

Combining (16), (17), and (18), we conclude that

$$\bar{\vartheta}_{1/2}(G_{n,p}) \geq \bar{\vartheta}_{1/2}(G_{n,p}) - 1 \geq \frac{\binom{n}{2}p - 8n^{3/2}p^{1/2}(1-p)^{1/2}}{(\lambda+4)n^{3/2}p^{1/2}(1-p)^{1/2}} \geq \frac{1}{2(\lambda+4)}\sqrt{\frac{np}{1-p}}$$

holds with probability at least $1 - \exp(-n)$. As $\bar{G}_{n,p} = G_{n,1-p}$, this proves (2) and the lower bounds in Thm. 4.

Remark 17. Suppose that $np = O(1)$. Then (2) shows that the probability that $\bar{\vartheta}_{1/2}(G_{n,p})$ is less than $c\sqrt{np}$ is exponentially small, for some constant $c > 0$. It is easily seen that no similar statement holds for the upper tail, i.e. for the event that $\bar{\vartheta}_{1/2}(G_{n,p}) > \zeta\sqrt{np}$ for some large ζ. The reason is that the probability that $\omega(G_{n,p}) > \zeta\sqrt{np}$ is at least $p^{\zeta^2 np} \geq \exp(-O(\ln(n))$, and $\bar{\vartheta}_{1/2}(G) \geq \omega(G)$ for all G.

4.2 Spectral Considerations

Let us briefly recall Juhász's proof that $\vartheta(G_{n,p}) \leq (2 + o(1))\sqrt{n(1-p)/p}$ for constant values of p, say. Given a graph $G = (V, E)$, we consider the matrix $M = M(G) = (m_{ij})_{i,j=1,\dots,n}$, where

$$m_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \notin E \\ (p-1)/p & \text{otherwise,} \end{cases} \qquad (i \neq j), \qquad (19)$$

and $m_{ii} = 1$ for all i. Then $\lambda_1(M) \geq \vartheta(G)$. Moreover, as p is constant, the result of Füredi and Komlos [20] on the eigenvalues of random matrices applies and yields that $\vartheta(G_{n,p}) \leq \lambda_1(M) \leq (2 + o(1))\sqrt{n(1-p)/p}$ whp. This argument carries over to the case $\ln(n)^7/n \leq p \leq 1/2$:

Lemma 18. *Suppose that* $\ln(n)^7/n \leq p \leq 1/2$. *Then* $\|M(G_{n,p})\| \leq 3\sqrt{n/p}$ *whp.*

Proof. As it is assumed in [20] that the variance of the matrix entries is independent of n, the proof of Füredi and Komlos [20] needs some minor adaptions to prove the lemma; all details have been carried out in [7, Sec. 4]. □

However, it is easily seen that in the sparse case, e.g. if $np = O(1)$, we have $\lambda_1(M) \gg n$ whp. The reason is that in the case $np \geq \ln(n)^7$ the random graph $G_{n,p}$ is "almost regular", which is not true if $np = O(1)$ (cf. [32]). We will get around this problem by chopping off all vertices of degree considerably larger than np, as first proposed in [1]. Thus, let $\varepsilon > 0$ be a small constant, and consider the graph $G' = (V', E')$ obtained from $G = G_{n,p}$ by deleting all vertices of degree greater than $(1 + \varepsilon)np$.

Lemma 19. *Suppose that* $c_0/n \leq p \leq \ln(n)^7/n$ *for some large constant* c_0. *Let* $G = G_{n,p}$, *and let* $M' = M(G')$. *Then* $\mathrm{P}(\|M'\| \leq c_1\sqrt{n/p}) \geq 9/10$, *where* $c_1 > 0$ *denotes some constant.*

To prove L. 19, we make use of the following lemma, which is implicit in [16, Sections 2 and 3].

Lemma 20. *Let $G = G_{n,p}$ be a random graph, where $c_0/n \leq p \leq \ln(n)^7/n$ for some large constant $c_0 > 0$. Let $n' = \#V(G')$, $e = n'^{-1/2}\mathbf{1} \in \mathbf{R}^{n'}$, and $A' = A(G')$. For each $\delta > 0$ there is a constant $C(\delta) > 0$ such that in the case $np \geq C(\delta)$ with probability $\geq 1 - \delta$ we have*

$$\max\{|\langle A'v, e\rangle|, |\langle A'v, w\rangle|\} \leq c_1\sqrt{np} \text{ for all } v, w \perp \mathbf{1},\ \|v\| = \|w\| = 1. \tag{20}$$

Here $c_1 > 0$ denotes a certain constant.

In addition, the proof of Lemma 19 needs the following observation.

Lemma 21. *Let c_1 be a large constant. The probability that in $G = G_{n,p}$ there exists a set $U \subset V$, $\#U \geq n/2$, such that $|\#E(G[U]) - \#U^2p/2| \geq c_1(\#U)^{3/2}p^{1/2}$ is less than* $\exp(-n)$.

Proof. There are at most 2^n sets U. By Chernoff bounds (cf. [26, p. 26]), for a fixed U the probability that $|\#E(G[U]) - \#U^2p/2| \geq c_1(\#U)^{3/2}p^{1/2}$ is at most $\exp(-2n)$, provided that c_0, c_1 are large enough. □

Proof of Lemma 19. Let $G = G_{n,p}$, let $n' = \#V(G')$, and let A', e be as in L. 20. Without loss of generality, we may assume that $V' = V(G') = \{1, \ldots, n'\}$. Let $c_1 > 0$ be a sufficiently large constant. Let J signify the $n' \times n'$ matrix with all entries equal to 1. Letting $\delta > 0$ be sufficiently small and $c_0 \geq C(\delta)$, we assume in the sequel that (20) holds, and that G has the property stated in L. 21. Let $z \in \mathbf{R}^{n'}$, $\|z\| = 1$. Then we have a decomposition $z = \alpha e + \beta v$, $\|v\| = 1$, $v \perp \mathbf{1}$, $\alpha^2 + \beta^2 = 1$. Since $\|M'z\| \leq \|M'e\| + \|M'v\|$, if suffices bound $\max_{v \perp e, \|v\|=1} \|M'v\|$ and $\|M'e\|$.

Let $\rho : \mathbf{R}^{n'} \to \mathbf{R}^{n'}$ be the projection on the space $\mathbf{1}^\perp$. Then $A'v = \rho A'v + \langle A'v, e\rangle e$, whence $\|A'v\| \leq \|\rho A'v\| + c_1\sqrt{np}$, for all unit vectors $v \perp \mathbf{1}$. In order to bound $\|\rho A'v\|$, we estimate $\|\rho A'\rho\|$ via (20):

$$\|\rho A'\rho\| = \sup_{\|y\|=1} |\langle \rho A'\rho y, y\rangle| = \sup_{\|y\|=1} |\langle A'\rho y, \rho y\rangle| = \sup_{\|y\|=1,\ \mathbf{1} \perp y} |\langle A'y, y\rangle| \leq c_1\sqrt{np}.$$

Consequently, $\|M'v\| = \|(J - \frac{1}{p}A')v\| = \frac{1}{p}\|A'v\| \leq 2c_1\sqrt{n/p}$ for all unit vectors $v \perp \mathbf{1}$.

To bound $\|M'e\|$, note that $-pM' = A' - pJ$. Let $\bar{d} = 2\#E(G')/n'$, and $x = A'e - (\bar{d}/n')Je$. Then $x \perp \mathbf{1}$, and by (20) we have $\|x\|^2 = \langle A'e, x\rangle - \langle(\bar{d}/n')Je, x\rangle = \langle A'e, x\rangle \leq c_1\sqrt{np}\|x\|$, whence $\|x\| \leq c_1\sqrt{np}$. By L. 21, $|\bar{d} - n'p| \leq c_1\sqrt{np}$. As a consequence, $\|(\bar{d}/n')Je - pJe\| \leq c_1\sqrt{np}$. Therefore, $\|pM'e\| \leq \|x\| + \|(\bar{d}/n')Je - pJe\| \leq 2c_1\sqrt{np}$, i.e. $\|M'e\| \leq 2c_1\sqrt{n/p}$. □

4.3 Bounding $\vartheta_2(G_{n,p})$ from above

Let $c_0/n \leq p \leq 1/2$ for some large constant $c_0 > 0$. First we observe that the largest eigenvalue of the matrix $M(G)$ considered in the previous section provides an upper bound on $\vartheta_2(G)$. (Actually this follows from the characterization of $\bar{\vartheta}_2$ as an eigenvalue minimization problem given in [39]. However, as [39] does not contain the proof, we show a brief ad hoc argument.)

Lemma 22. *Let G be any graph. Let $M = M(G)$. Then $\lambda_1(M) \geq \bar{\vartheta}_2(G)$.*

Proof. Let $\lambda > \lambda_1(M)$. Then the matrix $\lambda E_n - M$ is positive definite, whence there exist vectors $b_1, \ldots, b_n \in \mathbf{R}^n$ such that $m_{ij} = -\langle b_i, b_j\rangle$ for $i \neq j$, and $\lambda - 1 = \lambda - m_{ii} = \langle b_i, b_i\rangle = \|b_i\|^2 > 0$. Let $a_i = (\lambda - 1)^{-1/2}b_i$. Then $\|a_i\| = 1$ for all i. Moreover, if $i \neq j$ and $\{i,j\} \notin E$, then $\langle a_i, a_j\rangle = m_{ij}/(\lambda - 1) = -1/(\lambda - 1)$. If $\{i,j\} \in E$, then $\langle a_i, a_j\rangle \geq 0$. Hence $(a_1, \ldots, a_n)$ is a rigid vector λ-coloring of G. Therefore, $\vartheta_2(G_{n,p}) \leq \lambda$ for all $\lambda > \lambda_1(M)$. □

In the case $\ln(n)^7/n \le p \le 1/2$, combining L. 18 and L. 22 yields that $\vartheta_2(G_{n,p}) \le c_2\sqrt{n/p}$ whp. for some constant $c_2 > 0$, as desired. Thus, let us assume that $c_0/n \le p \le \ln(n)^7/n$ in the sequel. Let $\varepsilon > 0$ be a small constant.

Lemma 23. *With probability at least $9/10$ the random $G_{n,p}$ has at most $1/p$ vertices of degree greater than $(1+\varepsilon)np$.*

Proof. For each vertex v of $G_{n,p}$, the degree $d(v)$ is binomially distributed with mean $(n-1)p$. By Chernoff bounds (cf. [26, p. 26]), the probability that $d(v) > (1+\varepsilon)np$ is at most $\exp(-\varepsilon^2 np/100)$. Hence, the expected number of vertices v such that $d(v) > (1+\varepsilon)np$ is at most $n\exp(-\varepsilon^2 np/100) < 1/(10p)$, provided $np \ge c_0$ for some large constant $c_0 > 0$. Therefore, the assertion follows from Markov's inequality. $\square$

Let $G = G_{n,p}$, and let $G' = (V', E')$ be the graph obtained from G by deleting all vertices of degree greater than $(1+\varepsilon)np$. Let $V'' = V \setminus V'$, and $G'' = G[V'']$. Combining L. 23 and L. 19, we obtain that $\mathrm{P}\left(\vartheta_2(G') \le c_2\sqrt{n/p} \text{ and } \vartheta_2(G'') \le \#V(G'') \le 1/p \le \sqrt{n/p}\right) > 1/2$, where c_2 denotes a suitable constant. Consequently, Prop. 10 yields that

$$\mathrm{P}(\vartheta_2(G_{n,p}) \le (c_2+1)\sqrt{n/p}) > 1/2.$$

Let $\mu = (c_2+1)\sqrt{n/p}$, $t = \ln(n)\sqrt{n}$, and note that $t = o(\sqrt{n/p})$. Then, by Thm. 1,

$$\mathrm{P}(\vartheta_2(G_{n,p}) > \mu + t) \le 30\exp\left(-\Omega(\ln(n)^2)\right) = o(1).$$

Since $t < \sqrt{n/p}$, we get that $\vartheta_2(G_{n,p}) \le (c_2+2)\sqrt{n/p}$ with high probability.

4.4 Bounding $\bar{\vartheta}_2(G_{n,p})$ from Above

Let us first assume that $\ln(n)^7/n \le p \le 1/2$. Let $G = (V,E) = G_{n,p}$ be a random graph, and consider the matrix $\bar{M} = \frac{1}{1-p}E_n - \frac{p}{1-p}M(G)$, where E_n is the $n\times n$-unit matrix, and $M(G)$ is the matrix defined in (19). Combining L. 18 and L. 22, we have

$$\bar{\vartheta}_2(G) \le \lambda_1(\bar{M}) \le \|\frac{1}{1-p}E - \frac{p}{1-p}M'\| \le \frac{p}{1-p}\|M\| + 2 \le c_4\sqrt{np}$$

whp., where $c_4 > 0$ is a certain constant.

Now let $c_0/n \le p \le \ln(n)^7/n$ for some large constant $c_0 > 0$. In this case, the proof of our upper bound on $\bar{\vartheta}_2(G_{n,p})$ relies on the concentration result Thm. 3.

Lemma 24. *Whp. the random graph $G = G_{n,p}$ admits no set $U \subset V$, $\#U \le 1/p$, such that $\chi(G[U]) > \sqrt{np}$.*

Proof. We shall prove that for all $U \subset V$, $\#U = \nu \le 1/p$, we have $\#E(G[U]) < \nu\sqrt{np}/2$. Then each subgraph $G[U]$ has a vertex of degree $< \sqrt{np}$, a fact which immediately implies our assertion. Thus, let $\nu \le 1/p$. The probability that there exists some $U \subset V$, $\#U = \nu$, $\#E(G[U]) \ge \nu\sqrt{np}/2$, is at most

$$\binom{n}{\nu}\binom{\binom{\nu}{2}}{\nu\sqrt{np}/2}p^{\nu\sqrt{np}/2} \le \left(\frac{\mathrm{e}n}{\nu}\left(\frac{\mathrm{e}\nu^2 p}{\nu\sqrt{np}}\right)^{\sqrt{np}/2}\right)^{\nu} = \left(\frac{\mathrm{e}n}{\nu}\left(\frac{\mathrm{e}\nu\sqrt{p}}{\sqrt{n}}\right)^{\sqrt{np}/2}\right)^{\nu}$$

Let $b_\nu = (\mathrm{e}n/\nu)(\mathrm{e}\nu\sqrt{p}/\sqrt{n})^{\sqrt{np}/2}$. Observe that the sequence $(b_\nu)_{\nu=1,\dots,n}$ is monotone increasing, and that $b_{1/p} = \mathrm{e}np(\mathrm{e}/\sqrt{np})^{\sqrt{np}/2} \le \exp(-2)$. Therefore,

$$\sum_{\nu=\ln(n)}^{1/p} b_\nu^\nu \le b_{1/p}^{\ln(n)}/p \le n^{-2}p^{-1} = o(1).$$

Moreover, if $\nu \leq \ln(n)$, then $b_\nu \leq \mathrm{e}n\nu^{-1}(\mathrm{e}\nu\sqrt{p}/\sqrt{n})^{\sqrt{np}/2} \leq 1/n$, whence $\sum_{\nu=1}^{\ln n} b_\nu^\nu = o(1)$. Thus, $\sum_{\nu=1}^{1/p} b_\nu^\nu = o(1)$, thereby proving the lemma. □

Let $G = (V, E) = G_{n,p}$ be a random graph, and let $G' = (V', E')$ be the graph obtained from G by removing all vertices of degree greater than $(1+\varepsilon)np$, where $\varepsilon > 0$ is small but constant. Let $V'' = V \setminus V'$, and let $G'' = G[V'']$. By L. 23, with probability at least 9/10 we have $\#V'' \leq 1/p$. Therefore, by L. 24,

$$\mathrm{P}(\bar{\vartheta}_2(G'') \leq \sqrt{np}) \geq \mathrm{P}(\chi(G'') \leq \sqrt{np}) \geq 9/11.$$

To bound $\bar{\vartheta}_2(G')$, we consider the matrix $\bar{M} = \frac{1}{1-p}E_{n'} - \frac{p}{1-p}M(G')$, where $E_{n'}$ is the $\#V' \times \#V'$-unit matrix, and $M(G')$ the matrix (19). By L. 22, $\bar{\vartheta}_2(G') \leq \lambda_1(\bar{M})$. Moreover, by L. 19, with probability $\geq 9/10$ we have

$$\bar{\vartheta}_2(G') \leq \lambda_1(\bar{M}) \leq \|\frac{1}{1-p}E - \frac{p}{1-p}M'\| \leq \frac{p}{1-p}\|M'\| + 2 \leq c_4\sqrt{np},$$

for some constant $c_4 > 0$. Prop. 10 implies that $\bar{\vartheta}_2(G) \leq \bar{\vartheta}_2(G') + \bar{\vartheta}_2(G'')$, whence we conclude that $\mathrm{P}(\bar{\vartheta}_2(G_{n,p}) \leq (c_4+1)\sqrt{np}) > 1/2$. Since Thm. 3 shows that $\bar{\vartheta}_2(G_{n,p})$ is concentrated in width one, we have

$$\mathrm{P}\left(\bar{\vartheta}_{1/2}(G_{n,p}) \leq \bar{\vartheta}(G_{n,p}) \leq \bar{\vartheta}_2(G_{n,p}) \leq (c_4+1)\sqrt{np} + 1\right) = 1 - o(1),$$

thereby completing the proof of Thm. 4.

Remark 25. One could prove slightly weaker results on the probable value of $\vartheta(G_{n,p})$ and $\bar{\vartheta}(G_{n,p})$ than provided by Thm. 4 without applying any concentration results, or bounds on the SDP relaxation SMC of MAX CUT. Indeed, using only Lemmata 23, 24, 19 (thus implicitly [16]) and the estimates proposed in [27], one could show that for each $\delta > 0$ there is $C(\delta) > 0$ such the following holds. If $np \geq C(\delta)$, then

$$\mathrm{P}(c_1\sqrt{n/p} \leq \vartheta(G_{n,p}) \leq c_2\sqrt{n/p}) \geq 1 - \delta,\ \mathrm{P}(c_3\sqrt{np} \leq \bar{\vartheta}(G_{n,p}) \leq c_4\sqrt{np}) \geq 1 - \delta. \tag{21}$$

Such an approach is mentioned without proof independently in the latest version of [16] (the phrase "with high probability" is used in the sense "with probability $1 - o(1)$ as $np \to \infty$" in that paper). However, Thm. 4 is a bit stronger than (21), as the bounds (1) on $\vartheta(G_{n,p})$ and $\bar{\vartheta}(G_{n,p})$ hold with probability $1 - o(1)$ as $n \to \infty$ even if np remains bounded. Moreover, using the simpler approach it seems hard to obtain exponentially small probabilities as in (2).

4.5 The Lower Bound on $\mathrm{SDP}_k(G_{n,p})$

Having established Thm. 4, we know that there exist constants $c_0, c_1, c_2 > 0$ such that in the case $c_0/n \leq p \leq 1/2$ we have

$$c_1\sqrt{np} \leq \bar{\vartheta}_2(G_{n,p}) \leq c_2\sqrt{np} \tag{22}$$

with high probability. Let $k \geq 2$ be a fixed integer, and let us assume that $c_3k^2/n \leq p \leq 1/2$, where $c_3 = \max\{c_0, c_1^{-1}\}$. Let $G = G_{n,p}$ satisfy (22), and consider an rigid vector $\bar{\vartheta}_2(G)$-coloring $(v_1, \ldots, v_n)$ of G. Then

$$\langle v_i, v_j \rangle \geq -\frac{1}{c_1\sqrt{np} - 1} \geq -\frac{1}{k-1}$$

for all i, j, whence $(v_1, \ldots, v_n)$ is a feasible solution to SDP_k. Furthermore, if $\{i, j\} \in E$, then

$$\langle v_i, v_j \rangle \leq -\frac{1}{c_2\sqrt{np} - 1}.$$

Consequently, letting $A = (a_{ij})_{i,j=1,\ldots,n}$ be the adjacency matrix of G, we have

$$\mathrm{SDP}_k(G) \geq \sum_{i<j} a_{ij} \frac{k-1}{k}(1 - \langle v_i, v_j \rangle) \geq \left(1 - \frac{1}{k}\right) \#E(G) + \frac{\#E(G)}{2c_2\sqrt{np}}.$$

As $\#E(G_{n,p})$ is concentrated about its mean $\binom{n}{2}p$, we conclude that

$$\mathrm{SDP}_k(G_{n,p}) \geq \left(1 - \frac{1}{k}\right)\binom{n}{2}p + \frac{n^{3/2}p^{1/2}}{3c_2}$$

with high probability, thereby proving Cor. 5.

Remark 26. Consider the following relaxation SDP'_k of SDP_k:

$$\mathrm{SDP}'_k(G) = \max \sum_{i<j} a_{ij} \frac{k-1}{k} \left(1 - \langle v_i, v_j \rangle\right) \text{ s.t. } \|v_i\| = 1,$$

where $A = (a_{ij})_{i,j=1,\ldots,n}$ is the adjacency matrix of G and the max is taken over all families $v_1, \ldots, v_n$ of unit vectors in $\mathbf{R}^n$. Then $\mathrm{SDP}'_k(G) = (2(k-1)/k)\mathrm{SMC}(G)$. Consequently, L. 16 shows that $\mathrm{SDP}'_k(G_{n,p}) \leq (1 - 1/k)\binom{n}{2}p + c_3 n^{3/2}p^{1/2}$ whp., where $c_3 > 0$ is some constant. Thus, Cor. 5 implies that whp. both SDP_k and SDP'_k overestimate the weight $\mathrm{MC}_k(G_{n,p})$ of a MAX k-CUT by at least $c_4 n^{3/2}p^{1/2}$, for some constant $c_4 > 0$. Thus, in the case of random graphs the additional constraints $\langle v_i, v_j \rangle \geq -1/(k-1)$ only affect the precise constant in front of the second order term $n^{3/2}p^{1/2}$.

5 Random Regular Graphs

We show how to adapt the arguments given in the previous section to cover the case of random regular graphs. Throughout we assume that $r \geq c_0$ for some large constant $c_0 > 0$.

The proof of (3) relies on the upper bound on $\mathrm{SMC}(G_{n,r})$ [9, Thm. 15], and is similar to the proof of (2). To prove the upper bound on $\vartheta_2(G_{n,r})$, $c_0 \leq r = o(n^{1/4})$, we switch to the *configuration model* (cf. [37]). Let $W = V \times \{1, \ldots, r\}$. The elements of W are called *half edges*. A *configuration* ρ is a partition of W into $m = rn/2$ pairs, where we assume that rn is even. Thus, to each half-edge (u, v), ρ assigns another half-edge $\rho(u, v) \neq (u, v)$ such that $\rho^2 = \mathrm{id}$. We say that (u, v) and $\rho(u, v)$ form an *edge*. By $\mathcal{C} = \mathcal{C}(r)$ we denote the set of all configurations. Then $\#\mathcal{C}(r) = (2m-1)!!$.

To each $\rho \in \mathcal{C}$, the canonical map $\pi : W \to V$ assigns an r-regular multigraph $\pi(\rho)$. If we equip $\mathcal{C}$ with the uniform distribution, then conditional on $G_{n,r}$, π induces the uniform distribution. By $\sigma(\rho)$ we denote the simple graph obtained from the multigraph $\pi(\rho)$ by deleting all loops and turning all multiple edges into single edges. We define the *adjacency matrix* $A = A(\rho) = (a_{ij})_{i,j}$ of $\rho \in \mathcal{C}$ to be the matrix with entries

$$a_{ij} = \text{number of edges joining } i \text{ and } j \text{ in } \pi(\rho)$$

if $i \neq j$, and let a_{ii} be twice the number of loops at vertex i in $\pi(\rho)$.

Lemma 27. *Let $i \in \{1/2, 1, 2\}$. Let μ be the expectation of $\vartheta_i \circ \sigma$ over $\mathcal{C}$ (where $\vartheta_i \circ \sigma(\rho) = \vartheta_i(\sigma(\rho))$). Then for any $t > 0$ we have* $\mathrm{P}(|\vartheta_i \circ \sigma - \mu| > t) \leq 2\exp(-t^2/(128m))$.

Proof. If two configurations ρ, ρ' are such that $\sigma(\rho)$ can be transformed into $\sigma(\rho')$ by adding or deleting at most 4 edges, then $|\vartheta_i \circ \sigma(\rho) - \vartheta_i \circ \sigma(\rho')| \leq 4$ by Prop. 10. Therefore, the martingale argument used in the proof of Lemma 14 of [9] carries over without essential changes. □

The proof of the following lemma goes along the lines of [17]. However, as we work with the configuration model and consider also the case that the degree r tends to infinity, some adaptions are necessary; these have been carried out in [9].

Lemma 28. *There is a constant $\gamma > 0$ such that with high probability the adjacency matrix $A = A(\pi(\rho))$, $\rho \in \mathcal{C}$, satisfies $|\langle Ax, y\rangle| \leq \gamma\sqrt{r}$, for all unit vectors $x \perp \mathbf{1}$, $y \perp \mathbf{1}$.*

Given a configuration $\rho \in \mathcal{C}$, we let $M = M(\rho) = J - \frac{n}{r}A(\pi\rho)$. Then M is a symmetric $n \times n$ matrix.

Lemma 29. *There is a constant $c > 0$ such that whp. we have $\|M(\rho)\| \leq cnr^{-1/2}$.*

Proof. Let $M = M(\rho)$. As $M\mathbf{1} = 0$, it suffices to prove that $\langle Mv, w\rangle \leq cnr^{-1/2}$ for all unit vectors $v, w \perp \mathbf{1}$ whp. But this follows from L. 28 easily. $\square$

Lemma 30. *The expected number of loops and multiple edges in $\pi(\rho)$, $\rho \in \mathcal{C}$, is at most $9r^2$. Hence, with probability $\geq 1/2$ there are at most $18r^2$ loops or multiple edges.*

Proof. Let $(u,t), (u,s), (v,r), (v,r') \in W$ be distinct half edges. The probability that $\rho(u,t) = (v,r)$ and $\rho(u,s) = (v,r')$ is $\sim (2m)^{-2}$. There are $2m$ choices of (v,r), and then at most r choices of (v,r'). Further, given (u,t), there are r possible choices of s. Hence, the expected number of half edges that participate in multiple edges is at most $2(2m)^2r^2(2m)^{-2} \leq 8r^2$.

As for loops, let $(u,t) \in W$, and let $s \in \{1,\ldots,r\} \setminus \{t\}$. The probability that $\rho(u,t) = (u,s)$ is $\sim (2m)^{-1}$. Since there are at most $2m$ possible choices of (u,t), and then at most r choices of s, the expected number of loops is at most $2(2m)r(2m)^{-1} \leq 4r < r^2$. $\square$

Lemma 31. *There are constant $c_1, c_2 > 0$ such that whp. a random configuration $\rho \in \mathcal{C}$ satisfies $c_1nr^{-1/2} \leq \vartheta_{1/2}(\sigma\rho) \leq \vartheta(\sigma\rho) \leq \vartheta_2(\sigma\rho) \leq c_2nr^{-1/2}$.*

Proof. Let $\mathcal{B}$ be the event that the number of multiple edges and loops in $\pi\rho$, $\rho \in \mathcal{C}$, is at most $20r^2$. By L. 30, $\mathrm{P}(\mathcal{B}) \geq 1/2$. Consequently, by L. 29, there is a constant $c_1 > 0$ such that $\mathrm{P}(\|M(\rho)\| \leq c_1nr^{-1/2}|\mathcal{B}) \geq 1/2$. Hence, $\mathrm{P}(\rho \in \mathcal{B}$ and $\|M(\rho)\| \leq c_1nr^{-1/2}) \geq 1/4$.

We claim that if $\rho \in \mathcal{B}$ satisfies $\|M(\rho)\| \leq c_1nr^{-1/2}$, then $\vartheta_2(\sigma\rho) \leq 2c_1nr^{-1/2}$. For let Y be the set of all vertices $v \in V$ that participate in a multiple edge or a loop. Then $y = \#Y \leq 40r^2$. Relabeling the vertices if necessary, we may assume that $Y = \{n-y+1,\ldots,n\}$. Let $M = M(\rho) = (m_{ij})_{i,j=1,\ldots,n}$, and set $M' = (m_{ij})_{i,j=1,\ldots,n-y}$. Then

$$m_{ij} = \begin{cases} 1 & \text{if } i,j \text{ are non-adjacent in } \sigma\rho \\ (r-n)/r & \text{otherwise,} \end{cases} \qquad (1 \leq i < j \leq n-y),$$

and $m_{ii} = 1$ for all i. Let H be the simple graph on $V(H) = \{1,\ldots,n-y\}$ induced by $\sigma\rho$. Then $\vartheta_2(H) \leq \lambda_1(M') \leq \|M\| \leq c_1nr^{-1/2}$. Since the graph $\sigma\rho$ can be obtained from H by adding y vertices, and since $y \leq nr^{-1/2}$, we conclude that $\vartheta_2(\sigma\rho) \leq \vartheta_2(H) + y \leq 2c_1nr^{-1/2}$. Hence, $\mathrm{P}(\vartheta_2(\sigma\rho) \leq 2c_1nr^{-1/2}) \geq 1/4$. Invoking L. 27 completes the proof of the upper bound.

As for the lower bound, let c_2 be a sufficiently large constant, and let $\rho \in \mathcal{B}$ be such that the adjacency matrix $A = A(\pi(\rho))$ satisfies $|\langle A\xi, \eta\rangle| \leq c_2r^{1/2}$ for all unit vectors $\xi, \eta \perp \mathbf{1}$. Let Y, y, and H be as before, $y \leq 40r^2$. Moreover, let $\bar{A}$ be the adjacency matrix of $\sigma(\rho)[V \setminus Y]$, and let E_{n-y} and E_n be unit matrices of size $n-y$ and n. Since $\mathbf{1}$ is an eigenvector of $J - A + E_n$, we have

$$\lambda_n(J - A + E_n) = 1 - \lambda_2(A) = \max_{\xi \perp \mathbf{1},\ \|\xi\|=1} \langle A\xi, \xi\rangle \geq -c_2r^{1/2}.$$

Hence, $\lambda_n(\bar{A}) \geq -c_2r^{1/2}$, because $\bar{A} = J - A(\sigma(\rho)[V \setminus Y]) + E_{n-y}$ is a principal minor of $J - A + E_n$. Let $B = (n-y)^{-1}\left(E_{n-y} - \lambda_n(\bar{A})^{-1}\bar{A}\right)$. Then the matrix $B = (b_{ij})_{i,j=1,\ldots,n-y}$ is positive semidefinite, and we have $\sum_i b_{ii} = 1$. Moreover, $b_{ij} \geq 0$ for all i,j, and if $i,j \in V \setminus Y$

are adjacent in $\sigma(\rho)$, then $b_{ij} = 0$. It is shown in [23, pp. 51ff] that such a matrix B satisfies $\vartheta_{1/2}(\sigma(\rho)[V \setminus Y]) \geq \sum_{i,j} b_{ij}$ (the proof goes along the lines of [29, Sec. 7–9]). Hence,

$$\vartheta_{1/2}(\sigma(\rho)) \geq \vartheta_{1/2}(\sigma(\rho)[V \setminus Y]) \geq \sum_{i,j} b_{ij} \geq nr^{-1/2}/(2c_2).$$

Finally, applying L. 27 once more yields our assertion. □

It is shown in [37] that in the case $r = o(n^{1/4})$ we have $\mathrm{P}(\pi(\rho) \text{ is a simple graph}) \geq \exp(-o(n^{1/2}))$. Therefore, letting $t = \Omega(nr^{-1/2})$ and applying L. 27 we conclude that there is some constant $c > 0$ such that $\mathrm{P}(\vartheta_2(G_{n,r}) \leq cnr^{-1/2}) = 1 - o(1)$. A similar argument proves the lower bound on $\vartheta_{1/2}(G_{n,r})$.

6 Approximating the Independence Number and Deciding *k*-colorability

In this section we present the algorithms required for Thms. 6–8. The algorithm for the independent set problem is essentially identical with that proposed in [7], and the algorithm for deciding k-colorability resembles that given in [30]. Thus, our contribution is that using our new results on the Lovász number of random graphs and the vector chromatic number, we can improve on the *analyses* given in [7,30].

Approximating the independence number. The algorithm `ApproxMIS` for approximating the independence number consists of two parts. First, we employ a certain greedy procedure that on input $G = G_{n,p}$ most probably finds an independent set of size at least $\ln(np)/(2p)$, thereby providing a lower bound on $\alpha(G)$. Secondly, we compute $\vartheta(G)$ to bound $\alpha(G)$ from above. Throughout, we assume that $np \geq c_0$ for some large constant c_0.

Following [33], to find a large independent set of $G = G_{n,p}$, we run the greedy algorithm for graph coloring and pick the largest color class it produces. Remember that the greedy algorithm goes through the vertices $v = 1, \ldots, n$ of G, and assigns to v the least color among $\{1, \ldots, n\}$ that is not occupied by a neighbor $w < v$ of v.

Lemma 32. *The probability that the largest color class produced by the greedy coloring algorithm contains* $< \ln(np)/(2p)$ *vertices is at most* $\exp(-n)$.

Proof. The proof given in [33] for the case that $p \geq n^{\varepsilon - 1/2}$ carries over. □

The following algorithm is essentially identical with the one given in [7]. (The difference between the algorithm proposed in [33] and the one below is that our algorithm uses the Lovász number as an upper bound on $\alpha(G)$ instead of the largest eigenvalue of the matrix (19).)

Algorithm 33. `ApproxMIS`(G)
Input: A graph $G = (V, E)$. *Output:* An independent set of G.

1. Run the greedy algorithm for graph coloring on input G. Let I be the largest resulting color class. If $\#I < \ln(np)/(2p)$, then go to 5.
2. Compute $\vartheta(G)$. If $\vartheta(G) \leq C\sqrt{n/p}$, then output I and terminate. Here C denotes some sufficiently large constant (cf. the analysis below).
3. Check whether there exists a subset S of V, $\#S = 25\ln(np)/p$, such that $\#V \setminus (S \cup N(S)) > 12(n/p)^{1/2}$. If no such set exists, then output I and terminate.
4. Check whether in G there is an independent set of size $12(n/p)^{1/2}$. If this is not the case, then output I and terminate.
5. Enumerate all subsets of V and output a maximum independent set.

Lemma 34. *The expected running time of* ***ApproxMIS***$(G_{n,p})$ *is polynomial.*

Proof. The first two steps can be implemented in polynomial time, because $\vartheta(G)$ can be computed efficiently [24] (we may disregard rounding issues in this paper). By Thm. 4, the median μ of $\vartheta(G_{n,p})$ is at most $c\sqrt{n/p}$, for some constant c. Therefore, Thm. 1 entails that the probability that ApproxMIS runs step 3 is less than $\exp(-(n/p)^{1/2})$, provided C is large enough. Furthermore, up to polynomial factors, step 3 consumes time $\leq \exp(25\ln(np)^2/p) < \exp(\sqrt{n/p})$. Hence, the expected time spent executing step 3 is polynomial. Taking into account L. 32, the expected running time of the remaining steps can be estimated as in the proof of Thm. 4 in [7]. □

Finally, we claim that there exists some constant $\zeta > 0$ such that ApproxMIS(G) finds an independent set of size at least $\alpha(G)\ln(np)/(\zeta\sqrt{np})$, for all graphs G. Since step 5 will always find an independent set of size $\alpha(G)$, we may assume that $\#I \geq \ln(np)/(2p)$. Thus, if $\alpha(G) \leq C'\sqrt{n/p}$, then $\#I/\alpha(G) \geq \ln(np)/(2C'\sqrt{np})$. Further, it is easily seen that in the case $\alpha(G) > C'\sqrt{n/p}$ the algorithm ApproxMIS will run step 5.

Remark 35. The lower bounds on $\vartheta_{1/2}(G_{n,p})$ in Thm. 4 shows that we could not achieve an approximation ratio of $(np)^{\beta}$, $\beta < 1/2$, even if we would use the relaxation $\vartheta_{1/2}$ instead of ϑ to upper-bound the independence number in our algorithm ApproxMIS.

Deciding k-colorability. Following [30], we decide k-colorability by computing the vector chromatic number of the input graph. Let $k = k(n)$ be a sequence of positive integers such that $k(n) = o(\sqrt{n})$. Since the vector chromatic number is always a lower bound on the chromatic number, the answer of the following algorithm is correct for all input graphs G.

Algorithm 36. Decide$_k(G)$
Input: A graph $G = (V, E)$. *Output:* Either "$\chi(G) \leq k$" or "$\chi(G) > k$".

1. If $\bar{\vartheta}_{1/2}(G) > k$ then terminate with output "$\chi(G) > k$".
2. Otherwise, compute $\chi(G)$ in time $o(\exp(n))$ using Lawler's algorithm [34], and answer correctly.

Thm. 8 is a consequence of the following lemma.

Lemma 37. *Suppose that $p \geq Ck^2/n$ for some large constant C. Then the expected running time of* ***Decide***$_k(G^{+}_{n,p})$ *is polynomial.*

Proof. In [28] it is shown that $\bar{\vartheta}_{1/2}$ can be computed in polynomial time (we disregard rounding issues). Since the second step consumes time $o(\exp(n))$, inequality (2) shows that the expected running time of Decide$_k$ on input $G_{n,p}$ is polynomial. Consequently, by (5), we conclude that the expected running time of Decide$_k(G^{+}_{n,p})$ is polynomial. □

The analysis of Decide$_k$ on input $G_{n,r}$, $r \geq Ck^2$, is based on (3) and yields the proof of Thm. 9.

Remark 38. The analysis of Decide$_k$ shows that we can decide in polynomial expected time whether $G^{+}_{n,p}$ is k-colorable, provided $np \geq c_0k^2$. Conversely, the upper bounds on $\bar{\vartheta}_{1/2}(G_{n,p})$, $\bar{\vartheta}(G_{n,p})$, $\bar{\vartheta}_2(G_{n,p})$ in Thm. 4 show that even if we would use the relaxation $\bar{\vartheta}_2$ instead of the vector chromatic number $\bar{\vartheta}_{1/2}$ in our algorithm Decide$_k$, we would still have to assume that $np = \Omega(k^2)$.

7 Conclusion

The results presented in this paper show the Lovász number and other SDP relaxations of the independence number or the chromatic number provide powerful tools in the design of algorithms with a polynomial expected running time. Indeed, in addition to the algorithmic applications given in this paper, the results were used by Coja-Oghlan, Goerdt, Lanka, and Schädlich to obtain an algorithm for deciding in polynomial expected time whether a random $2k$-SAT formula is satisfiable [10]. Furthermore, using Thms. 1 and 4, Coja-Oghlan obtained an algorithm that finds a large independent set hidden in a semirandom graph in polynomial expected time [8]. Therefore, the author expects that the general idea of combining large deviation techniques such as Talagrand's inequality with SDP relaxations will lead to further contributions to the algorithmic theory of random structures (cf. also [9]).

In comparison with purely combinatorial techniques or computing eigenvalues, semidefinite programming requires a rather heavy machinery (cf. [24]). However, compared to the eigenvalues of the adjacency matrix, semidefinite programs such as the Lovász number seem to be rather "robust" (cf. the discussion in [13]). In fact, this robustness may be the reason why for the Lovász number we can derive a large deviation result such as Thm. 1. By contrast, for "small" values of p no similarly strong tail bounds on the eigenvalues of the auxiliary matrix $M(G_{n,p})$ are known (cf. [3,33])–although we used to the eigenvalues of $M(G_{n,p})$to bound the mean of $\vartheta(G_{n,p})$.

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